

Uncertainty Principle Consequences at Thermal Equilibrium

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Contrary to the conventional wisdom that deviations from standard thermodynamics originate from the strong coupling to the bath, it is shown that these deviations are intimately linked to the power spectrum of the thermal bath. Specifically, it is shown that the lower bound of the dispersion of the total energy of the system, imposed by the uncertainty principle, is dominated by the bath power spectrum and therefore, quantum mechanics inhibits the system thermal-equilibrium-state from being described by the canonical Boltzmann's distribution. This is in sharp contrast to the classical case, for which the thermal equilibrium distribution of a system interacting via central forces with pairwise-self-interacting environment, irrespective of the interaction strength, is shown to be *exactly* characterized by the canonical Boltzmann distribution. As a consequence of this analysis, we define an *effective coupling* to the environment that depends on all energy scales in the system and reservoir interaction. Sample computations in regimes predicted by this effective coupling are demonstrated. For example, for the case of strong effective coupling, deviations from standard thermodynamics are present and, for the case of weak effective coupling, quantum features such as stationary entanglement are possible at high temperatures.

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Introduction.—Thermodynamics was developed before the atomistic description of Nature was formulated. Statistical mechanics was then introduced to understand the laws of thermodynamics in terms of a microscopic description, thus closing the gap between macroscopic and microscopic description. Due to the interest in quantum technologies, there is a major ongoing effort to develop a consistent and well defined extension of thermodynamics to the quantum regime [1–3]. However, the majority of these theories are primarily based on Boltzmann's original ideas and are therefore plagued by issues concerning irreversibility, the origin of the second law, the relation between physics and information, the meaning of ergodicity, etc. (see, e.g., Ref. [4]).

Despite these issues, it is now well known that, e.g., Onsager's regression hypothesis fails in the quantum realm [5, 6] and that non-Markovian dynamics are relevant in a variety of fields and applications, from foundations [2, 7], to nuclear physics [8], quantum metrology [9, 10] and biological systems (see, e.g., [11] and references therein). It is also known that the thermal equilibrium state of a quantum system *strongly* coupled to a thermal bath deviates from the canonical Boltzmann distribution [1, 12–14], this is also expected to occur in the classical case [15]. Since both are incoherent thermal stationary situations, one would expect that the quantum system is devoid of any coherence and hence, based on the decoherence program [16], that both distributions should coincide. However, one might also suggest that the entanglement

between the system and the bath, which has no classical counterpart, could introduce quantum-classical deviations [1]. Furthermore, the fact that extra deviations could be present even if the entanglement between the system and the bath is zero [17] makes the situation even more intriguing.

Hence, it seems appropriate to find a situation where the classical and the quantum contributions to the deviation from the Boltzmann distribution can be clearly isolated and examined. Here we show that *irrespective of the interaction strength*, there are no deviations from Boltzmann's distribution when a classical system interacts via central forces with a pairwise-self-interacting environment. Thus, if after quantum-mechanically treating the same case, deviations from the canonical Boltzmann's distribution are present, then they are purely quantum in nature. As shown below, deviations do appear and, based on completely general arguments, are shown to rely on the *uncertainty principle* characteristic of quantum mechanics.

Therefore, the uncertainty principle not only inhibits the system's thermal-equilibrium-state from being described by the canonical Boltzmann distribution, but for *each system-bath interaction it also selects which thermal equilibrium states are physically accessible*. This latter remark, formulated here for the first time in the framework of quantum thermodynamics, constitutes the cornerstone of the theory of pointer states (the states which are robust against the presence of the environment) [16] and

could have deep consequences for an understanding of the thermalization of quantum systems.

Classical Thermal-Equilibrium-State.—To provide a well-defined situation, consider a classical particle of mass m with potential energy $U_S(q)$ and Hamiltonian $H_S(p, q) = \frac{1}{2m}p^2 + U_S(q)$. Now consider a collection of \mathfrak{N} classical particles interacting via the central force potential $U_B^{i,j}(\mathbf{q}_i - \mathbf{q}_j)$ and Hamiltonian $H_B(\mathbf{p}, \mathbf{q}) = \sum_j \mathfrak{N} [\frac{1}{2m_j} \mathbf{p}_j^2 + \sum_{i,j} U_B^{i,j}(\mathbf{q}_i - \mathbf{q}_j)]$, with which the classical system interacts via the central force potential energy $\mathcal{V}_j(\mathbf{q}_j - q)$, so that the total Hamiltonian is given by

$$H = H_S(p, q) + H_B(\mathbf{p}, \mathbf{q}) + \sum_j \mathfrak{N} \mathcal{V}_j(\mathbf{q}_j - q). \quad (1)$$

In classical statistical mechanics, the thermal equilibrium distribution of the system S is defined by

$$\rho_S(p, q) = \frac{1}{Z} \int \prod_j \mathfrak{N} d\mathbf{p}_j d\mathbf{q}_j \exp[-H(p, q, \mathbf{p}_j, \mathbf{q}_j)\beta], \quad (2)$$

where $Z = \int \prod_j \mathfrak{N} d\mathbf{p}_j d\mathbf{q}_j \int dp dq \exp(-H\beta)$ denotes the partition function of the total system, with $\beta = 1/k_B T$ and T being the temperature of the environment. The integral over \mathbf{p}_j in Eq. (2) trivially cancels out with the corresponding contribution in Z . Due to the particular dependence of \mathcal{V}_j and $U_B^{i,j}$ on \mathbf{q}_i , \mathbf{q}_j and q , the integrals over $\{\mathbf{q}_j\}$ can be appropriately manipulated, with the net result that they cancel out with the corresponding contribution in Z . Thus,

$$\rho_S(p, q) = Z_S^{-1} \exp[-H_S(p, q)\beta], \quad (3)$$

where $Z_S = \int \prod_j \mathfrak{N} d\mathbf{p}_j d\mathbf{q}_j \exp(-H_S\beta)$. Hence, the thermal equilibrium distribution of a bounded particle in contact with a pairwise-self-interacting thermal bath via central forces, *irrespective of the coupling strength*, is exactly given by the canonical Boltzmann distribution.

This result is surprising because, in the strong coupling regime, there is no apparent physical reason why the equilibrium thermodynamic properties of a system are independent, for a wide class of systems, of both the nature of the bath to which it is coupled and of the functional form of the observables that mediate the interaction. The physical picture that emerges from this result is that *in the long time regime*, any dissipative mechanism is equally effective in taking the system to thermal equilibrium. In other words, dissipative dynamics can contract the classical phase-space volume with no fundamental restriction and therefore, the resultant equilibrium state is independent of the dissipative coupling and the rate at which equilibrium [Eq. (3)] is reached. This suggests that the concept of intrinsic and extensive thermodynamic variables [4] can be extended, in some cases, to the strong coupling regime. By contrast, and as shown below, in the

quantum realm, dissipative mechanisms are accompanied by decoherence effects and are bath-nature and coupling-particularities sensitive [16, 18] and are then capable of inducing a variety different thermal states.

Quantum Thermal-Equilibrium-State.—The quantum description follows from Eq. (1) by the standard quantization procedure. Based on the general description given in [3, 13, 14], one can easily extend the classical definition in Eq. (2) to the quantum regime, namely,

$$\hat{\rho}_S = \frac{1}{Z} \text{tr}_B \exp \left\{ - \left[\hat{H}(\hat{p}, \hat{q}, \hat{\mathbf{p}}_j, \hat{\mathbf{q}}_j) \right] \beta \right\}. \quad (4)$$

The operator character of the various terms in Eq. (4) and their commutativity relations prevent us from proceeding as we did in the classical case. However, these very same commutativity relations allow the immediate formulation of the following set of inequalities,

$$[\hat{H}_S, \hat{V}] \neq 0 \Rightarrow \Delta \hat{H}_S \Delta \hat{V} \geq \frac{1}{2} |[\langle \hat{H}_S, \hat{V} \rangle]|, \quad (5a)$$

$$[\hat{V}, \hat{H}_B] \neq 0 \Rightarrow \Delta \hat{V} \Delta \hat{H}_B \geq \frac{1}{2} |[\langle \hat{V}, \hat{H}_B \rangle]|, \quad (5b)$$

where $\hat{V} = \sum_j \mathfrak{N} \mathcal{V}_j(\hat{\mathbf{q}}_j - \hat{q})$ and $\Delta \hat{O} = \sqrt{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}$ denotes the standard deviation of \hat{O} , with $\langle \hat{O} \rangle = \text{tr}(\hat{O} \hat{\rho})$, $\hat{\rho}$ being the thermal equilibrium state of the system S and the bath B.

Some implications follow from Eqs. (5). Specifically, since $|[\langle \hat{H}_S, \hat{V} \rangle]|$ is a measure of the quantum correlations between the system and the bath, it dictates the lower bound of $\Delta \hat{H}_S \Delta \hat{V}$. This lower bound is different for each interaction since each particular form of \hat{V} imposes a different commutation relation. This last statement is precisely what allows, for example, for a connection between the theory of pointer states and quantum thermodynamics. *The general bounds in Eqs. (5) predict different thermal equilibrium states for each type of interaction, a purely quantum effect.*

For example, since $[\hat{H}_S, \hat{V}] = 0$ implies a pure decohering interaction, which can be treated here in the framework of fluctuations without dissipation [19], the equilibrium state is an incoherent mixture of system's eigenstates and is expected to be well characterized by the canonical Boltzmann distribution [20]. In this case $[\hat{H}_S, \hat{V}] = 0$, so that $\Delta \hat{H}_S \Delta \hat{V} \geq 0$, meaning that the commutativity relation here results in the minimum lower bound on $\Delta \hat{H}_S \Delta \hat{V}$. Note that the same lowest limit is obtained if, as in the classical case, the thermal equilibrium state of the system $\hat{\rho}_S$ is formally the canonical Boltzmann distribution $\hat{\rho}_S^{\text{can}}$. Specifically, if $\hat{\rho} = \hat{\rho}_S^{\text{can}} \otimes \text{tr}_S \hat{\rho}$, then $|[\langle \hat{H}_S, \hat{V} \rangle]| = \text{tr}([\hat{H}_S, \hat{V}] \hat{\rho}) = \text{tr}([\hat{\rho}, \hat{H}_S] \hat{V}) = 0$ since $[\hat{\rho}_S^{\text{can}}, \hat{H}_S] = 0$, giving $\Delta \hat{H}_S \Delta \hat{V} \geq 0$. This is just a consequence of the fact that the Boltzmann distribution is the zero-order-in-the-coupling thermal equilibrium state and therefore, disregards quantum correlations between

the system and the bath. Below, it is shown that the bath spectrum is also related to the lower bound and thus, Eq. (5) will ALSO allow for a clear connection to other fundamental features such as the failure of the Onsager's regression hypothesis in the quantum regime [5, 6].

Influence of the Spectrum of the Bath: Non-Markovian Character at Thermal Equilibrium—Although the set of inequalities (5) are fairly general, it is not possible to infer the role that standard quantities such as the spectral density or the spectrum of the bath play in establishing the thermodynamic bounds above. To provide a concrete expression for the lower bound in Eq. (5a), the interaction between the bath particles is set to zero, i.e., $U_B^{i,j} = 0$ and consider the second order picture of the system-bath central force interaction, i.e., $\hat{V} \approx \sum_j \frac{1}{2} \mathbf{m}_j \omega_j^2 (\hat{q}_j - \hat{q})^2$, which yields to the well-known Ullersma-Caldeira-Leggett model [21, 22]. After expanding \hat{V} , it is possible to redefine \hat{H} in Eq. (1) as $\hat{H} = \hat{H}_S' + \hat{H}_B' + \hat{V}_{SB}$ with $\hat{V}_{SB} = \hat{B} \otimes \hat{S}$. Here $\hat{B} = \sum_j \mathbf{m}_j \omega_j^2 \hat{q}_j$ and $\hat{S} = \hat{q}$, which act in the Hilbert space of the bath and the system, respectively. The commutator $[\hat{H}_S, \hat{V}]$, calculated to second order in \hat{V}_{SB} , is then given by

$$|[\hat{H}_S, \hat{V}]| \propto \text{tr}_S \left\{ [\hat{H}_S, \hat{S}] e^{-\hat{H}_S \beta} \int_0^{\hbar\beta} d\sigma \hat{S}(-i\sigma) K(\sigma) \right\}, \quad (6)$$

where $\hbar K(\sigma) = \langle \hat{B}(-i\sigma) \hat{B}(0) \rangle_B$ denotes the two-time correlation of the bath operators given by [23] $K(\sigma) = \pi^{-1} \int d\omega J(\omega) \cosh(\frac{1}{2}\hbar\beta\omega - i\sigma) / \sinh(\frac{1}{2}\hbar\beta\omega)$, $J(\omega) = \pi \sum_j \frac{1}{2} \mathbf{m}_j \omega_j^3 \delta(\omega - \omega_j)$ being the spectral density of the bath. Note that as long as second order perturbation theory is valid, Eq. (6) holds for any \hat{S} and \hat{B} and can be straightforwardly generalized to the case of $\hat{V}_{SB} = \sum_\alpha \hat{B}_\alpha \otimes \hat{S}_\alpha$.

The main feature of the quantum thermodynamic bound in Eq. (6) is the presence of the power spectrum of the bath $I(\omega, T) = \hbar J(\omega) \coth(\frac{1}{2}\hbar\beta\omega)$, which for bare Ohmic dissipation, $J(\omega) = m\gamma\omega$, at high temperatures, $\hbar\beta \rightarrow 0$, is the flat $I(\omega, T) \approx 2m\gamma k_B T$. Note that in this high temperature limit, the upper limit of the integral in Eq. (6) vanishes, leading to the vanishing of the commutator, even if $[\hat{H}_S, \hat{S}] \neq 0$. A similar series-expansion analysis leads to the conclusion that the thermal equilibrium state $\hat{\rho}_S$ formally approaches the canonical Boltzmann distribution only when $\hbar\beta \rightarrow 0$. In other words, in the high temperature limit the quantum correlations between the bath and the system disappear and the thermal equilibrium state is described by the canonical distribution, irrespective of the coupling strength or the functional form of the spectral density $J(\omega)$. It is clear that these results cast doubt on arguments in favor of canonical typicality in the quantum regime [24, 25] at other than high temperatures.

For out-of-equilibrium quantum dynamics, the low temperature condition, finite $\hbar\beta$, is associated with non-

Markovian dynamics [23, 26]. Since at fixed T , this non-Markovian character can be modified by the functional form of the spectral density [27], Eq. (6) makes clear that the equilibrium system properties depend on the non-Markovian character. This means that the quantum equilibrium statistical properties of a system experiencing Markovian dynamics (flat spectrum) are expected to differ from those of the same system experiencing non-Markovian dynamics (non-flat spectrum), which is in sharp contrast to the classical case (see Eq. 3). This can be clearly understood in terms of the different thermodynamic lower bounds resulting from either Markovian or non-Markovian interactions [see Eq. (6)].

To make a connection with previous studies, the failure of Onsager's regression hypothesis in quantum mechanics [5, 6] is discussed next. In doing so, note that the hypothesis that knowing all mean values suffices to determine the quantum dynamics of the correlation functions is valid only under Markovian dynamics [5, 6] and when correlations between the bath and the system are negligible at equilibrium (in general, at any time) [28]. Based on the fact that formal Markovian dynamics can only be achieved for flat spectra (bare Ohmic spectral density with $\hbar\beta \rightarrow 0$ [27]), these two conditions can be seen as a single one when formulated in terms of Eq. (6). Specifically, Markovian dynamics imply $|[\hat{H}_S, \hat{V}]| \rightarrow 0$ and, hence, the system-bath correlations vanish. This implies that Onsager's regression hypothesis, as well as the Boltzmann distribution, pertains exclusively to the classical realm.

To provide some insight into the magnitude and consequences of the fundamental limit derived above and, in particular, of the role of the spectral density, an effective coupling to the bath is introduced below and sample aspects of two complementary regimes are analyzed. Specifically, (i) effective strong coupling characterized by deviations from standard thermodynamics, and (ii) an effective weak coupling that is shown below to allow for the survival of entanglement between two oscillators in thermal equilibrium at high temperatures.

Effective Coupling to the Bath.—For the Ullersma-Caldeira-Leggett model, a standard calculation [23], after removing a local contribution in the correlation function, yields $K(\tau) = \frac{2m}{\hbar\beta} \frac{d}{d\tau} \sum_{l=1}^{\infty} \tilde{\gamma}(|\nu_l|) \sin(|\nu_l|\tau)$, where $\nu_l = l\Omega$, with $\Omega = 2\pi/\hbar\beta$, are the Matsubara frequencies and $\tilde{\gamma}(z)$ defines an effective coupling to the bath. Note that $\tilde{\gamma}(|\nu_l|)$ contains all the information about the correlations of the bath operators and therefore defines the influence of the bath on the system at thermal equilibrium.

For the subsequent discussion we adopt the most commonly used spectral density, the regularized Drude model with a high frequency cutoff ω_D , $J(\omega) = m_0\gamma\omega\omega_D^2/(\omega^2 + \omega_D^2)$, where γ is the standard strength coupling constant to the thermal bath and ω_D dictates the degree of non-Markovian dynamics. For this particular case, the effective coupling is given by $\tilde{\gamma}(|\nu_l|) =$

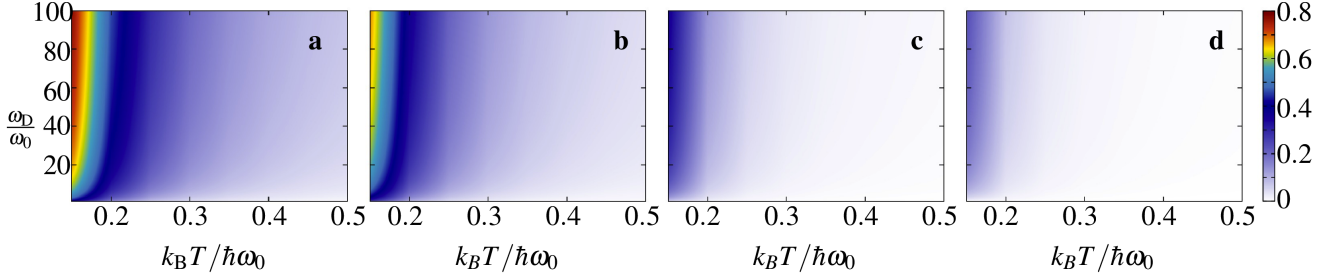


FIG. 1. $\log(Z/Z_{\text{can}})$ for a harmonic oscillator as a function of the ratios $k_B T / \hbar \omega_0$ and ω_D / ω_0 . We compare the partition function for $\gamma = 0.1\omega_0$ (a), $\gamma = 0.05\omega_0$ (b), $\gamma = 0.01\omega_0$ (c) and $\gamma = 0.005\omega_0$ (d). We observe that (i) the stronger the damping rate γ , the stronger the deviation from the canonical partition function, (ii) the larger the cutoff frequency, the larger the deviation from the canonical partition function and (iii) in the high temperature regime $\Omega/\omega_D \gg 1$, regardless the damping rate, no deviations are obtained.

$\gamma/(1 + |\Omega/\omega_D|)$. Below we analyze the effective weak coupling, $\Omega/\omega_D \gg 1$, and the effective strong coupling, $\Omega/\omega_D \ll 1$, regimes.

Strong Effective Coupling Regime—To quantify the consequences of non-flat spectra in this regime, consider as the system a harmonic oscillator of mass m_0 and frequency ω_0 coupled to a thermal bath [12, 13]. In particular, we are interested in quantifying: (i) the generation of squeezing in the thermal equilibrium state, (ii) the deviation from the canonical partition function $Z_{\text{can}} = \text{tr} e^{\hat{H}_{\text{S}}\beta}$ and (iii) the deviation from the canonical von-Neumann entropy $S_{\text{can}} = \text{tr} [\hat{\rho}_{\text{can}} \ln(\hat{\rho}_{\text{can}})]$.

For this case the momentum and position variances are given by [12, 13] $\langle p^2 \rangle = m_0^2 \omega_0^2 \langle q^2 \rangle + \Delta$, where $\langle q^2 \rangle = \langle q_{\text{cl}}^2 \rangle + \frac{2}{m_0 \beta} \sum_{n=1}^{\infty} [\omega_0^2 + \nu_n^2 + \tilde{\gamma}(|\nu_n|)|\nu_n|]^{-1}$ and the squeezing parameter $\Delta = -2m_0 \gamma \beta^{-1} \partial \ln Z' / \partial \gamma$ with $Z' = \frac{1}{\hbar \beta \omega} \prod_{n=1}^{\infty} |\nu_n| [\omega_0^2 + \nu_n^2 + \tilde{\gamma}(|\nu_n|)|\nu_n|]^{-1}$. We recall that for this model, the classical theory predicts $\langle p_{\text{cl}}^2 \rangle = m_0^2 \omega_0^2 \langle q_{\text{cl}}^2 \rangle$ and $\langle q_{\text{cl}}^2 \rangle = k_B T / m_0 \omega_0^2$, so that $\Delta_{\text{cl}} = 0$.

For the effective weak coupling regime $\Omega/\omega_D \gg 1$, disregarding terms of the order ω_0/ω_D and γ/ω_D gives $\pi \hbar \gamma m \omega_D / 6 \Omega$ [12]. Thus Δ vanishes at high temperatures, and the classical unsqueezed state is recovered. However, for the strong coupling regime $\Omega/\omega_D \ll 1$, $\Delta \approx \hbar \gamma m \ln(2\pi \omega_D / \Omega)$ [12], meaning that the deviation from the canonical state translates into squeezing of the equilibrium state. This feature may be of relevance toward the generation of non-classical states, e.g., in nano-mechanical resonators.

Deviations from the canonical result are also evident in the partition function Z . Figure 1 shows the logarithmic ratio of Z to the canonical partition function Z_{can} as a function of the dimensionless parameters $k_B T / \hbar \omega_0$ and ω_D / ω_0 for (from left to right) $\gamma = 0.1\omega_0$, $\gamma = 0.05\omega_0$, $\gamma = 0.01\omega_0$ and $\gamma = 0.005\omega_0$. Deviations are observed at low temperatures and for high cutoff frequencies (i.e., in the effective strong coupling regime). In the opposite limit, regardless of the coupling parameter γ , both calculated partition functions show the same behavior, as expected

from the discussion above. For the von Neumann entropy $S = \text{tr} [\hat{\rho}_{\text{S}} \ln(\hat{\rho}_{\text{S}})]$, the behavior of the ratio $\log(S/S_{\text{can}})$ is essentially the same as the one described for the partition function ratio in Fig. 1, and is not shown here.

Since $\tilde{\gamma}(z) = \frac{1}{m_0} \int_0^{\infty} \frac{d\omega}{\pi} \frac{J(\omega)}{\omega} \frac{2z}{\omega^2 + z^2}$, the use of different spectral densities changes the functional form of the effective coupling and therefore, of the thermal equilibrium properties. Hence, as long as $\hbar \beta$ remains finite, different spectral densities lead to different thermal equilibrium states.

Weak Effective Coupling Regime—As an example in the weak effective coupling regime, consider the survival of entanglement at thermal equilibrium between two identical harmonic oscillators with masses m_0 and frequencies ω_0 linearly coupled with coupling constant c_0 . The Hamiltonian is given by $\hat{H} = \hat{H}_{\text{S}} + \sum_{j,\alpha} \left[\frac{\hat{p}_{j,\alpha}^2}{2m_j} + \frac{m_j \omega_j^2}{2} (\hat{q}_{j,\alpha} - \hat{q}_\alpha)^2 \right]$, with $\alpha = \{1, 2\}$ and $\hat{H}_{\text{S}} = \frac{1}{2m_0} (\hat{p}_1^2 + \hat{p}_2^2) + \frac{1}{2} m_0 \omega_0^2 (\hat{q}_1^2 + \hat{q}_2^2) - c_0 q_1 q_2$. The introduction of independent baths for each oscillator ensures that no deviations from Boltzmann's distribution are present in the classical case. This can be verified directly from the multi-particle-system generalization of Eq. (1).

At equilibrium, the entanglement between the two harmonic oscillators can survive only when $k_B T / \hbar \omega_0 \ll 1$ (see, e.g., Ref. [29]). However, this limit only applies in the Markovian regime and $\gamma \rightarrow 0$. Thus, based on the discussion above, and supported by the recent observation that non-Markovian dynamics assists entanglement in the longtime limit [9], we expect that this limit needs to be refined in order to account for the non-Markovian character of the interaction and the finite value of γ .

Figure 2 shows the logarithmic negativity for different values of the damping constant γ as a function of the dimensionless ratios $k_B T / \hbar \omega_0$ and ω_D / ω_0 . As expected, (i) the more coupled the oscillators are, the higher the temperatures and the damping rate at which entanglement can survive at equilibrium (not shown), and (ii) the smaller the damping rate (the more isolated the system is), the higher the temperature at which entanglement

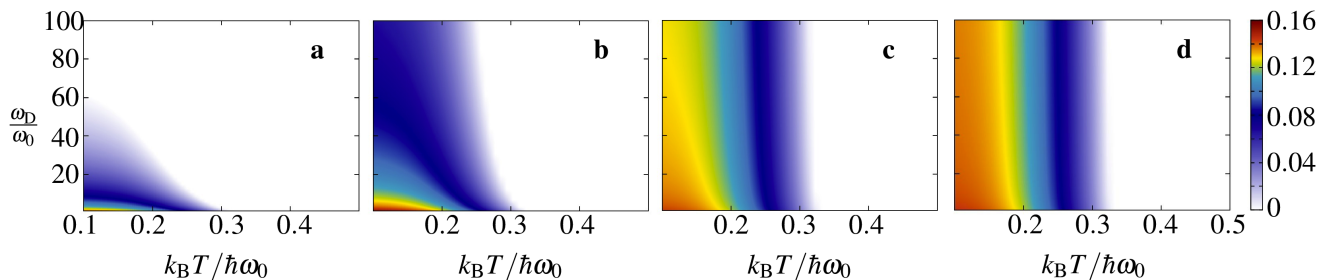


FIG. 2. Logarithmic Negativity in the presence of non-Markovian interactions for $c_0 = 0.1 m_0 \omega_0^2$ with $\gamma = 0.1 \omega_0$ (a), $\gamma = 0.05 \omega_0$ (b), $\gamma = 0.01 \omega_0$ (c) and $\gamma = 0.005 \omega_0$ (d) as a function of the dimensionless parameters $k_B T / \hbar \omega_0$ and ω_D / ω_0 . We observe that (i) the stronger the coupling between the oscillators, c_0 , the higher the temperature at which the system can support entanglement at equilibrium (not shown); (ii) the higher the damping rate γ , the stronger the influence of the non-Markovian interaction depicted by the ratio ω_D / ω_0 ; and (iii) the smaller the ratio ω_D / ω_0 (a more non-Markovian interaction), the higher the temperature at which the system can support entanglement at equilibrium.

can be maintained. The new feature here is that the more non-Markovian the interaction, the higher the temperature and the damping rate at which entanglement can be maintained at equilibrium.

Discussion.—It has been shown that the fundamental bound imposed by the Heisenberg uncertainty principle [Eq. (6)] prevents quantum systems to relax the Gibbs state dictated by the system Hamiltonian only. The Gibbs state is only recovered in the classical-high T limit ($\hbar\beta \rightarrow 0$). The implications at low- T for quantum thermodynamics are so crucial as the failure for the Onsager hypothesis or the difficulty on the definitions of the specific heat [30] and the temperature definition [31]. Specifically, the high temperature regime defined by $\hbar\beta \rightarrow 0$, modified by an appropriate effective coupling, emerges as the main condition for the vanishing of those deviations. Although the concrete examples presented here are particular for the second order approximation of the interaction potential, the general picture provided here remains valid, albeit more involved, in non-linear cases and with non-Gaussian statistics. The results presented here clarify the role of non-Markovian dynamics and its relevance at thermal equilibrium, and provide physical insights into how non-Markovian interactions contribute to protecting quantum features such as entanglement. Moreover, they may shed light on the role of non-Markovian dynamics and their effects in the derivation of fundamental limits in areas such as quantum metrology [10] and quantum speed limits [7].

The result also serves to evaluate other measures of quantum correlation. For example, the logarithmic negativity used in Ref. [17] to quantify the system-bath entanglement is found wanting. That is, it vanishes although there are still deviations from the canonical state. As such this measure is unable to quantify the quantum correlations and that other kind of quantum correlation witnesses, such as discord, are required.

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